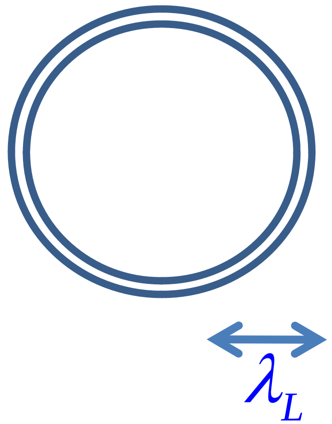
Lecture 3A

The Josephson effect: « semiclassical » considerations

1.Flux quantization in a superconducting ring

Let's start by revisiting the topic of the Meissner effect.As in lecture 2A,let's imagine a superconducting ring whose thickness d is much less than its mean radius R,and also much less than any other relevant lengths such as the London penetration depth,d<< λ_L .

Recall that the eigenfunction χ_0 of $\hat{\rho}_2$ is of the general form $\chi_0(r_1 r_2 \sigma_1 \sigma_2) \equiv \chi_0(r_1 - r_2, \sigma_1 \sigma_2 : R)$



where R is the center-of-mass coordinate.For a simple BCS (s-wave) superconductor the dependence on the spins and on the relative coordinate is fixed,so we fix these at some arbitrary« reference » values,let us say $r_1 - r_2 = 0, \sigma_1 = -\sigma_2 = \uparrow$,and concentrate on the dependence on the **center-of-mass** coordinate,which we now rename r: $\chi_0(r_1 r_2 \sigma_1 \sigma_2)_{r_1 = r_2, \sigma_1 = -\sigma_2 = \uparrow} \equiv \Psi(r)$ (This quantity is essentially the Ginzburg – Landau order parameter).

Suppose we had a single particle described by a Schroedinger wave function $\psi(rt)$:Then we have for the (single-particle) density and current (in the absence of a magnetic vector potential) the standard expressions:

$$\rho(rt) = |\psi(rt)|^2, \; j_{mass}(rt) = (-i\hbar / m)(\psi^* \nabla \psi - c.c)$$

So,if we write $\psi(rt)$ in terms of its amplitude and phase:

$$\psi(rt) = |\psi(rt)| \exp i\phi(rt)$$

then we have

$$\rho(rt) = |\psi|^2, \; j(rt) = |\psi|^2 \nabla\phi(rt)$$

So can define a quantity with the dimensions of velocity:

$$v(rt) \equiv j = (\hbar / m)\nabla\phi(rt)$$

Next,consider a system of bosons (e.g.He-4) with BEC. Write the condensate wave function $\chi_0(rt)$ in the form $\chi_0(rt) = A(rt)\exp i\phi(rt)$ then the density and mass current of the condensate are given by ,

$$\rho_c(rt) \equiv N_0(t) |\chi_0(rt)|^2,$$

$$j_c(rt) \equiv (-i\hbar/2m) N_0(t)(\chi_0^*(rt)\nabla\chi_0(rt) - c.c.) = (\hbar/m) N_o(t) A^2(rt)\nabla\phi(rt)$$

So we can define a « condensate velocity » $v_c(rt)$ by

$$v_c(rt) \equiv j(rt) / \rho(rt) = (\hbar / m)\nabla\phi(rt)$$

(note the condensate number has dropped out!).For historical reasons,in the He-4 literature this quantity is actually usually called the « superfluid velocity » and denoted $v_s(rt)$. It follows immediately from the definition of $v_s(rt)$ that it satisfies

$$\nabla \times v_s(rt) = 0,$$

$$\oint_C v_s(rt) \bullet dl = nh / m$$

where n must be 0 if the contour C lies in a simply connected region,but otherwise may be a nonzero integer.

How much of the liquid is associated with the superfluid velocity?Answer:at T=0, all of it!I.e. at T=0,

$$j(rt) = \rho(rt)v_s(rt)$$

(more generally,coefficient is « superfluid density » $\rho_s(rt)$)

(in 4-He at T=0,condensate fraction is only ~10%,but $\rho_s = \rho$)

(c) BEC of diatomic molecules (neutral):

Associated mass is now 2m (m=fermion mass),so write

$$\rho_c(rt) \equiv |\psi(rt)|^2, \; j_c(rt) = (-i\hbar / 2(2m))(\psi^* \nabla \psi - c.c)$$

with $\psi(rt)$ now the wave function of the molecular center of mass. Again define

$$\psi(rt) \equiv A(rt) \exp i\phi(rt),$$

$$v_c(rt) \equiv v_s(rt) \equiv j_c(rt) / \rho_c(rt) = (\hbar / 2m) \nabla \phi(rt)$$

We still have $\nabla \times v_s = 0,$

but now $\oint_C v_s \bullet dl = nh / 2m$

Again,at T=0

$$j(rt) = \rho(rt) v_s(rt)$$

So again,condensate « drags » the rest of the liquid with it.

(d) Cooper pairs ,neutral:as BEC of diatomic molecules,but « condensate fraction » now very small $(\sim 10^{-4})$

(e) Cooper pairs in superconductors:

The only complication is that the electrons are electrically charged.For a single electron moving in an EM vector potential A(rt),the expression for the probability density is unmodified,

$$\rho(rt) = |\psi(rt)|^2$$

but the expression for the probability current is modified:

$$j(rt) = (1/2m)(-i\hbar\psi^*\nabla\psi - eA(rt) + H.c.)$$

and so the velocity now has an extra term:

$$v(rt) = (\hbar/m)(\nabla\phi(rt) - eA(rt)/\hbar)$$

For a BEC of diatomic molecules composed of two charged fermions,and also for Cooper pairs of electrons in a superconductor,the argument goes through as above,with the difference that since the relevant wave function $\psi(rt)$ is that of the center of mass,not only is the mass m replaced by 2m as above but the appropriate charge to put in is not e but 2e,so the pair velocity (« superfluid velocity ») $v_s(rt)$ is given by

$$v_s(rt) = (\hbar/2m)(\nabla\phi(rt) - 2eA(rt))$$

(most fundamental eqn.of macroscopic theory of superconductivity)

Once again,the probability current,and thus the electric current,associated with the superfluid velocity is ,at T=0,the total number (charge) density:writing explicitly the electric current,

$$j_{el}(rt) = n(rt)ev_s(rt) \cong nev_s(rt)$$

(n=mean density of electrons)

Under conditions where n(rt) is well approximated by n (equilibrium density of electrons) we can combine the two fundamental equations to give

$$j_{el}(rt) = (ne\hbar / 2m)(\nabla\phi(rt) - 2eA(rt)/\hbar) \quad (1)$$

Consequences of (1):(a)since

$\nabla \times \nabla\phi(rt) \equiv 0,$ we have for any geometry

$$\nabla \times j_{el}(rt) = -(ne^2/m)\nabla \times A(rt) = -(ne^2/m)B(rt) \quad (2)$$

(note that not only $\hbar$,but factor of 2 has fallen out!).Specializing to the time-independent case,we can combine this with the relevant Maxwell equation

$$\nabla \times H = \mu_0^{-1}\nabla \times B = j_{el}(r)$$

to give the result we obtained previously:

$$-\nabla \times (\nabla \times j_{el}(r)) = \nabla^2 j_{el}(r) = \lambda_L^{-2} j_{el}(r), \lambda_L \equiv (\mu_0 m / ne^2)^{1/2}$$

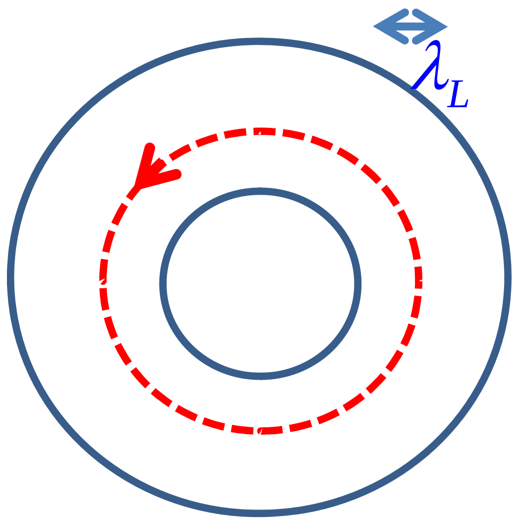
(since $\nabla \bullet j_{el}(rt) = \partial\rho(rt)/\partial t = 0$ in time-independent case)

Thus,electric current (and magnetic field) is exponentially small in a (3D) bulk super-conductor at distances >>λ_L from the surface,independently of the topology.

(b)However,in a multiply connected topology,the fact that B(r)vanishes does not imply that A(r) necessarily does!In fact,eqn.(2) is perfectly compatible with a contribution to eqn.(1) from an irrotational term which does not contribute to (2). (In a simply connected topology such as a sphere,such a term is not allowed as,since B(r) is zero everwhere in the bulk ,it would violate Stokes' theorem).But now eqn.(1) has an interesting consequence:

Flux quantization in a bulk superconducting ring:

We know (a)current vanishes on any path which circles the ring at a distance >> λ_L from either surface (b)quite generally,



$$j_{el}(r) = (ne\hbar / 2m)(\nabla\phi(r) - 2eA(r)/\hbar)$$

Thus,quite generally,deep inside a bulk superconducting ring (or indeed far from the surface in any geometry) we must have

$$\nabla\phi(r) = 2eA(r)/\hbar$$

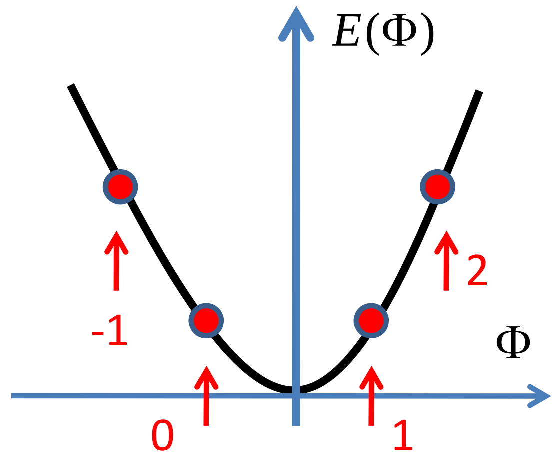
(where $\phi(r)$ is the phase of the Cooper-pair order parameter and A(r) the electromagnetic vector potential).For a simple ring as in the figure,we can integrate This equation around the ring:since $\phi(r)$ must be single-valued modulo 2π ,we get for the flux Φ trapped through the ring

$$\Phi \equiv \oint_C A(r)\cdot dl = (\hbar/2e)\oint_C \nabla\phi(r)\cdot dl = (\hbar/2e)\cdot 2n\pi \equiv n\Phi_0$$

where $\Phi_0 \equiv h/2e$ is the superconducting flux quantum.Thus,the flux through a thick superconducting ring is quantized in units of h/2e $(\sim 2\cdot 10^{-15}Tm^2 or 2\cdot 10^{-7}Gcm^2)$

How does this happen physically?Suppose we apply through the hole an arbitrary flux Φ_{ext} (in general not equal to an integral number of flux quanta).The system will respond by generating a Meissner current on the internal surface (i.e.the edge of the hole) of just sufficient a magnitude to cancel the difference between the applied flux and the nearest quantized value.Note for future reference that this process costs a self-inductance energy inductance energy equal to $LI^2/2 = (\Phi_{ext} - n\Phi_0)^2/2L$ where L is the self-inductance of the ring and Φ_{ext} the externally applied flux.

Let's consider the special case $\Phi_{ext} = \Phi_0 / 2.$ Then the groundstate is doubly degenerate;the two states correspond respectively to total trapped flux $\Phi = 0 or \Phi_0$ and circulating current $I = \pm\Phi_0 / 2L,$ hence energy $E = \Phi_0^2 / 8L$ in each case.The question arises:could these two states be possibly used as the basis states of a qubit? Certainly,they are well separated in energy from the next lowest pair of states $(\Phi = 2\Phi_0, -\Phi_0),$ which have flux different by $(\pm 3/2)\Phi_0$ from the externally applied value and hence energy higher by $\Phi_0^2 / L.$ On the other hand,there is a serious problem in using this basis as a qubit:the energy barrier which must be crossed to make a transition between them is enormous.The reason is that to pass between the two quantized values of the flux one must pass through unquantized values,and to relax the quantization condition the ring has to become normal. The energy cost (the superconducting condensation energy density times the volume) is of order $\Delta^2 N(0)V$ where Δ is the BCS energy gap;for reasonable values of the parameters (say sample dimensions a few microns) this is of the order of an MeV.(In addition,going through the normal phase would almost certainly inevitably generate heavy dissipation). Thus,one wouldlike to find some means of keeping the two « quantized » states while drastically lowering the barrier between them.



One way to do this is by interrupting the ring with a region where superconductivity is strongly suppressed-a Josephson junction.

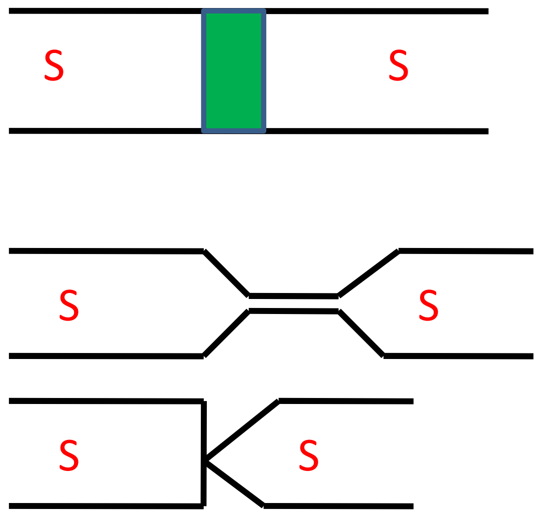
Josephson junctions

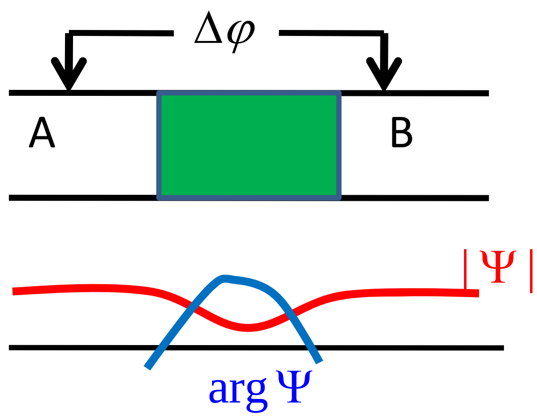
A Josephson junction (J.J) may be defined as any region separating two bulk superconducting regions,in which superconductivity is strongly suppressed.Some types of J.J.:tunnel oxide barrier,microbridge,point contact…

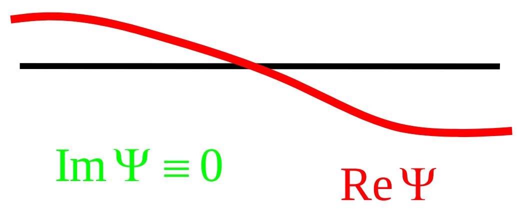
The crucial property of a J.J. is that the phase difference $\Delta\phi$ of the pair wave function $\Psi(r)$ across it may be varied away from zero without costing energies of a « bulk » order of magnitude.Suppose we start with $\Delta\phi = 0$ and gradually increase $\Delta\phi$ .As we do so,the amplitude Ψ of will decrease somewhat so as to minimize the sum of the « volume » and « bending » energies.When we come to to the point $\Delta\phi = \pi$,there are two possibilities:

(A)The amplitude remains everywhere nonzero at this point.Then the phase $\phi(r)$ remains meaningful throughout the junction,and the phase difference $\Delta\phi \equiv \int \nabla\phi(r)\cdot dl$ is defined in the range $[-\infty,\infty]$;the energy is obviously not a periodic function of $\Delta\phi$ .This is the « hysteretic » case, and is of little interest in the context of qubits.

(B)The amplitude may tend to zero at some point in the junction.In this case it is clear that values of $\Delta\phi$ differing by 2π are equivalent,so that the energy associated with the junction must be a unique periodic function of $\Delta\phi$.This is the nonhysteretic case,which is usually realized in tunnel-oxide junctions and is of interest for quantum computing.







Energetics of a Josephson junction

As we have seen,by definition ,the phase difference $\Delta\phi$ of the Cooper-pair wave function across a hysteretic J.J. is defined only modulo 2π ,and the energy must therefore be a periodic function of it.Moreover,time-reversal invariance implies that it must be even under $\Delta\phi \to -\Delta\phi$,so the most general form is

$$E(\Delta\phi) = \sum A_n \cos(n \cdot \Delta\phi)$$

However,in a tunnel-oxide barrier it can be shown that if the single-electron tunnelling matrix element is t,then the n-th term in the sum is of order t^{2n} .Hence,for a « weak » junction (small t) only the lowest nontrivial term need be kept;writing the coefficient as $-E_J$,we have

$$E(\Delta\phi) = -E_J \cos\Delta\phi$$ (canonical form of Josephson energy)

The quantity E_J is usually positive (so that the minimum energy occurs for $\Delta\phi = 0$),and we will assume this in what follows.To obtain a numerical value of E_J we need a specific model of the junction in question;for a tunnel-oxide junction of the type originally considered by Josephson,the classic calculation of Ambegaokar and Baratoff gives

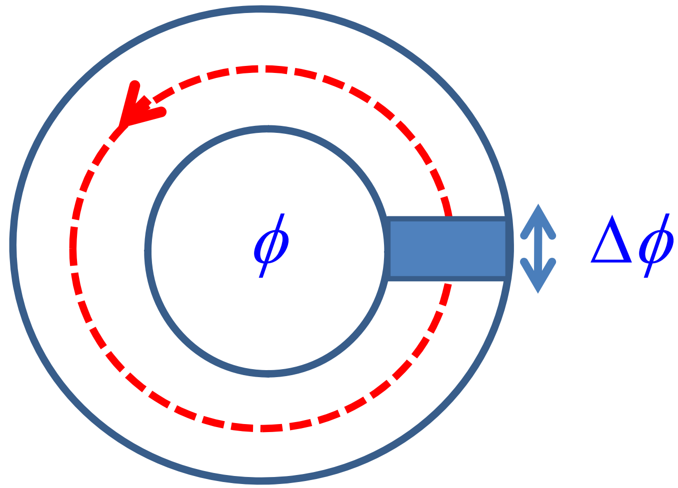
$E_J = \pi\Delta / 2eR_N \leftarrow$ resistance of junction when bulk metals in normal state

This form seems consistent with experiment on most simple junctions.However,one can equally well regard E_J as a phenomenological parameter to be fitted from experiment, and this is often done when the junction is used as (part of) a qubit.

Energetics of a SQUID ring (superconducting ring containing a JJ)

In addition to the energy $E_J(\Delta\phi)$ associated with the Josephson junction,a SQUID ring has a self-inductance energy $LI^2/2$ associated with the current I circulating in it.Since the total flux Φ is a sum of the externally applied flux Φ_{ext} and the contribution LI from the circulating current,we can express this term as a function of Φ : $E_{ind}(\Phi) = (\Phi - \Phi_{ext})^2 / 2L$

However, the variables $\Delta\phi$ and Φ are not independent:recall that for any path deep in a bulk superconductor,we have the relation



$$\nabla\phi(r) = 2eA(r)/\hbar$$

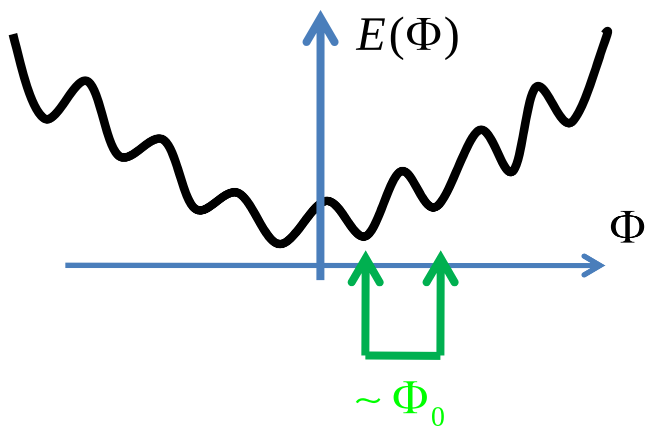
Integrating this relation along the path shown and using the fact that the contribution to $\int_C A\cdot dl$ from the part of the path which goes through the junction itslef is negligible,we obtain

$$\Delta\phi = 2\pi\Phi/\Phi_0$$

Thus,the total (« potential ») energy associated with a SQUID ring is a unique function of the total trapped flux Φ:

$$E(\Phi) = (\Phi - \Phi_{ext})^2 / 2L - E_J \cos(2\pi\Phi/\Phi_0)$$

In general this expression has a number of meta-stable minima separated by potential barriers.



The current though a Josephson junction

Let's consider the condition for (stable or metastable) equlibrium of the SQUID ring, $\partial E / \partial \Phi = 0.$: in explicit form this reads

$$(\Phi - \Phi_{ext})/L + (2\pi E_J / \Phi_0)\sin(2\pi\Phi / \Phi_0) = 0 \qquad (\%)$$

or,since as we have seen $LI = \Phi - \Phi_{ext}$ and $\Delta\phi = 2\pi\Phi / \Phi_0$,

$$I = (-2\pi E_J / \Phi_0)\sin\Delta\phi \qquad (*)$$

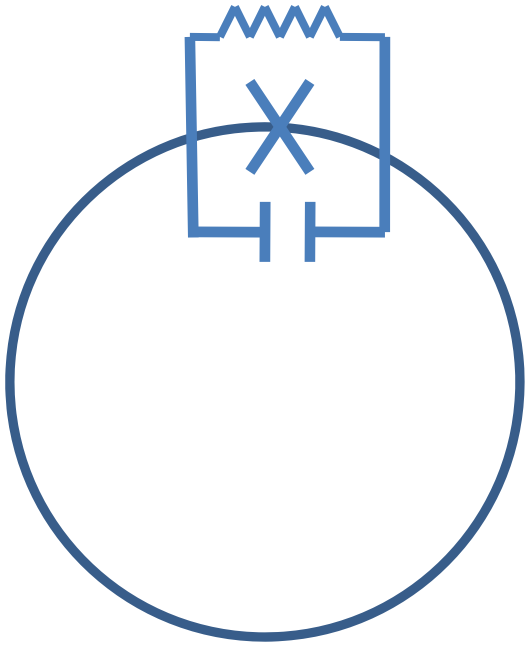
(-sign of no significance,simply reflects convention for $\Delta\phi$ relative to Φ)

Thus,in equilibrium (where the current flowing in the ring must equal that through the junction) the current through the junction is equal to $(-)I_c \sin\Delta\phi$ where the quantity $I_C \equiv 2\pi E_J / \Phi_0$ is the « critical current »(maximum supercurrent) of the junction. The relation (*) can be derived by other methods;since it relates the current uniquely to the phase drop,we may take it to be valid also under nonequilibrium conditions. We can now use this result to derive a more general dynamics for the SQUID.

Dynamics of a SQUID ring

Consider now a state of the ring where the trapped flux Φ does not correspond to a stable or metastable minimum,so that condition (%) is not fulfilled.This is equivalent to the statement that the current flowing in the bulk ring is not equal to that flowing across the junction (which is still given by the expression (*):note that the relation $\Delta\phi = 2\pi\Phi / \Phi_0$ is valid even in a time-dependent state,provided the relevant frequencies are <<the bulk plasma frequency).Where can the extra current go? If for the moment we neglect the possibility of the junction carrying a « non-Josephson » current,the only possibility is into the capacitance shunting the junction.

We conclude,then,that in a nonequilibrium situation there is a build-up of charge on the «plates » of the « capacitor » formed by the junction.Suppose this charge is Q,then the associated voltage is Q/C,where C is the capacitance of the junction (a quantity which we can rarely calculate from first principles,but may be able either to estimate or to take from experiment). Now the voltage V developed across the junction is just the rate of change of flux through the ring*: $V = -\partial\Phi / \partial t$



Since the extra current flowing into the capacitance is the difference between that in the bulk ring and that flowing through the junction,we have

Φ

$$(\Phi - \Phi_0)/L + I_C \sin(2\pi\Phi/\Phi_0) = dQ/dt = C(dV/dt)$$

or setting $V = -\partial\Phi / \partial t$ and rearranging,

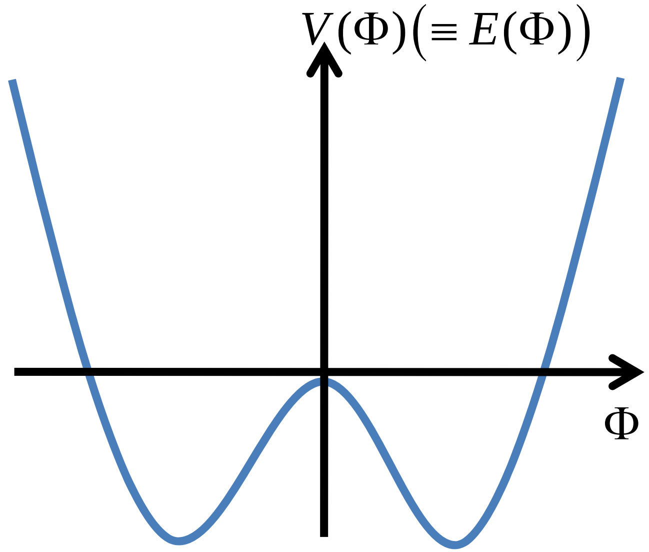
$$C(d^2\Phi/dt^2) + (I_C \sin(2\pi\Phi/\Phi_0 + (\Phi - \Phi_{ext})/L = 0$$

Finally,we need to take into account that in addition to the supercurrent $I_C \sin\Delta\phi$,the junction may be able to carry a « normal » current given by the standard Ohm's-law formula $I = V/R$.
Adding this to the current balance and expressing V in terms of the flux as above, we finally find the following equation for the total trapped flux in the SQUID ring:

$$C(d^2\Phi/dt^2) + (d\Phi/dt)/R + (I_C \sin(2\pi\Phi/\Phi_0) + (\Phi - \Phi_{ext})/L = 0$$

This is known as the «RSJ(C) » (resistively shunted junction with capacitance)model of a SQUID;it is evidently analogous to a particle moving in the « potential » E(Φ),with « mass » C .Note that so far we have treated the total trapped flux as an essentially classical variable:The only place QM comes in is in the determination of the various energies involved.

The special case (most) relevant to quantum computing



Recall that in general the « potential » in which the flux variable moves is given by

$$V(\Phi)(\equiv \grave{E}(\Phi)) = (\Phi - \Phi_{exzt})^2 / 2L - (I_C \Phi_0 / 2\pi)\cos(2\pi\Phi / \Phi_0)$$

A particularly interesting case occurs when the external flux is close to $\Phi_0 / 2$ and the dimensionless parameter $\beta_L \equiv 2\pi L I_C$ is just greater than 1. then the potential has the « quadratic-plus-quartic » shape shown:

$$V(\Phi) \equiv V(x) = -\alpha x^2 + \beta x^4 - \gamma x,$$
$$x \equiv \Phi - \Phi_0 / 2$$

where if V is measured in units of $\Phi_0^{\,2} / 2L$ and x in units of Φ_0,

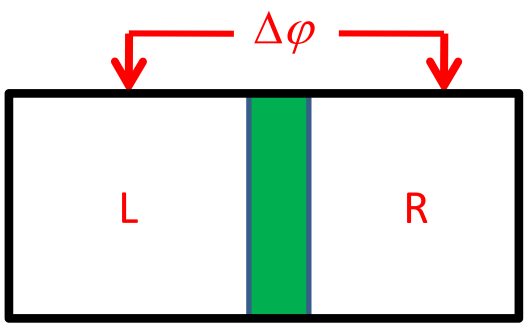
$$\alpha = \beta_L - 1, \beta = \beta_L / 6, \gamma = 2(\Phi_{ext} - \Phi_0 / 2)$$

The coefficients α and β are determined by the intrinsic SQUID parameters,while the coefficient γ ,which governs the asymmetry of the potential (the offset between the two classical minima) is controlled by the externally applied flux.Indeed,it may be easily shown that the positions x_0 of the minima,and their energy splitting ΔE ,are given by the approximate formulae

$$x_0 = \pm(6(\beta_L - 1))^{1/2}, \Delta E = 2|\Phi_{ext} - \Phi_0 / 2| \cdot \Phi_0 / L$$

These results will be essential when we come to consider the effects of QM on the behavior of the SQUID.

We finally need to consider two other geometries involving J.J's which may be used as qubits.
The conceptually simpler of the two is the « Cooper-pair box » geometry shown:the pair of boxes is isolated from the rest of the world,so the total number of electrons on it is conserved.In this geometry we may take the wave function of the COM of the Cooper pairs to have the schematic form



$$\Psi(r) \cong a(t)\Psi_L(r) + b(t)\Psi_R(r)$$

where the (real) wave function $\Psi_L(r)(\Psi_R(r))$ is localized on the left (right) of the junction.The amplitude of the coefficients a and b will in general depend on the normalization of $\Psi_{L(R)}$ and is not of great interest;what is of much more interest is the quantity

$$\Delta\phi(t) \equiv \arg(a(t)/b(t))$$

which is the phase drop across the J.J.As in the case of a junction in a SQUID ring,we may reasonably assume that the supercurrent through the junction is related to $\Delta\phi(t)$ by the equation

$$I(t) = I_C \sin\Delta\phi(t) \qquad (*)$$

and that the energy associated with a given value of $\Delta\phi(t)$ is,as in that case,given by

$$E(\Delta\phi) = -E_J \cos(\Delta\phi), E_J \equiv 2\pi I_C / \Phi_0$$

In the literature,eqn.(*) is often called the first Josephson equation.

To complete the dynamics,we need an equation for the evolution of the phase difference $\Delta\phi(t)$.
For this it is helpful to refer back to the SQUID case.There,the flux $\Phi(t)$ trapped in the circuit was related to $\Delta\phi(t)$. by $\Phi(t)=(\Phi_0/2\pi)\Delta\phi(t)$, so Faraday's law implies

$$d(\Delta\phi(t))/dt = 2eV(t)/\hbar \qquad (\#)$$

where V(t) is the voltage developed around the circuit,and hence equals the voltage drop across the junction.It is therefore plausible that eqn.(#) should apply equally to the « box » situation (Another derivation is given in lecture 3B).Eqn.(#) is called the « second Josephson equation » in the superconductivity literature.

We can now combine the two Josephson equations and use the fact that in the absence Of any normal current through the junction the supercurrent must be equal to the time rate Of change of the charge imbalance across the junction,which in turn is related to the voltage V(t) by the capacitance of the junction (mutual capacitance of the two boxes):

$$I(t) = I_C \sin(\Delta\phi(t)) = dQ(t)/dt = CdV(t)/dt = (C\Phi_0/2\pi)d^2(\Delta\phi(t)/dt^2$$

Or rewritten in terms of E_J ,

$$d^2\Delta\phi/dt^2 + (E_J/C)\sin(\Delta\phi) = 0$$

This is the analog of the SQUID equation in the absence of any dispative normal current.
As in that case,we can incorporate also a normal current given by V(t)/R;the effect is to add To the RHS a term $(\Phi_0/2\pi R)d\Delta\phi/dt$

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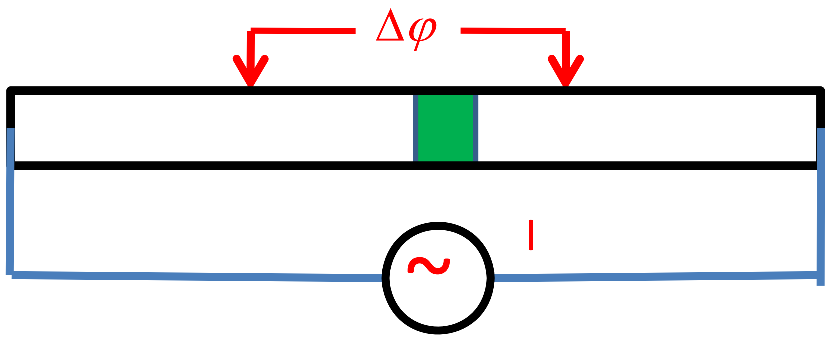
The final geometry we need to consider is that of a J.J.in an open circuit with a constant input current :



The analysis in this case goes through much as in the case of the Cooper-pair box;the only difference is that the current-balance equation contains an extra term due to the externally input current $I_{ext}(t)$. The « capacitance » which is now relevant is that of the junction itself,so we find (neglecting the normal current)

$$I_C \sin\Delta\phi(t) = I_{ext}(t) + CdV(t)/dt$$

which when combined with the second Josephson equation gives

$$d^2\Delta\phi(t)/dt^2 + (E_j/C)\sin\Delta\phi(t) = (2\pi/C\Phi_0)I_{ext}(t)$$

Note that in a steady state we have simply

$$\sin\Delta\phi = I_{ext}/I_C$$

i.e.the phase difference is controlled by the (constant) external current;since it is time-independent,no voltage is developed across the junction and no normal current flows (nor does any charge accumulate in the junction capacitance).