Quantum phases of strongly interacting bosons on a two-leg Haldane ladder

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(Received 30 October 2017; published 23 March 2018)

We study the ground-state physics of a single-component Haldane model on a hexagonal two-leg ladder geometry with a particular focus on strongly interacting bosonic particles. We concentrate our analysis on the regime of less than one particle per unit cell. As a main result, we observe several Meissner-like and vortex-fluid phases, both for a superfluid as well as for a Mott-insulating background. Furthermore, we show that for strongly interacting bosonic particles, an unconventional vortex-lattice phase emerges, which is stable even in the regime of hard-core bosons. We discuss the mechanism for its stabilization for finite interactions by a means of an analytical approximation. We show how the different phases may be discerned by measuring the nearest- and next-nearest-neighbor chiral currents as well as their characteristic momentum distributions.

DOI: 10.1103/PhysRevA.97.033619

I. INTRODUCTION

With the rapid progress in the realization of synthetic magnetism in ultracold atomic gases during recent years, experiments in this field are now at the cusp of complementing the theoretical approaches and solid-state experiments on topological effects in strongly correlated quantum systems [1,2]. At the same time, seminal advances in experiments with irradiated graphene [3,4] or photonic lattices [5–7] have shown the availability of these technologies for the investigation of topological states of matter as well. So far, experiments have succeeded with several proof-of-concept measurements of various topological effects, most of which, however, studied noninteracting particles. Among these efforts, we mention the quantum engineering of various Hofstadter-Harper–like models [8] with staggered [9,10] or rectified fluxes [11,12] in ultracold atoms. Highly nontrivial properties can be measured in these experiments, such as chiral currents [13–15], Chern numbers [16–18], or Berry curvatures [19–21]. The theoretical understanding of interaction effects of such models, however, remains challenging and has triggered numerous studies in this field. For Hofstadter-Harper–like models those include, e.g., predictions of interacting (fractional) Chern insulators [22–27] and other unconventional quantum states [28–33].

Another paradigmatic example of a model with nontrivial topological phases is the famous Haldane model [34], given by the Hamiltonian

$$H_H = -J \sum_{\langle \ell, \ell' \rangle} (c_\ell^\dagger c_{\ell'} + \text{H.c.}) - J_H \sum_{\langle \ell, \ell' \rangle} (e^{i \phi_{\ell \ell'}} c_\ell^\dagger c_{\ell'} + \text{H.c.}),$$

(1)

where $c_\ell$ ($c_\ell^\dagger$) describes a single-component fermionic or bosonic annihilation (creation) operator, with $\langle \ell, \ell' \rangle$ denoting nearest neighbors and $\langle \langle \ell, \ell' \rangle \rangle$ next-nearest neighbors. A sketch of the model is shown in Fig. 1. Contrary to the example of topological states of matter realized in an electronic system with a strong magnetic field [8], here, no net flux pierces the unit cell of the lattice and, hence, translational symmetry is not explicitly broken. In spite of its apparent complexity, i.e., the need of complex next-nearest-neighbor exchange terms, which seemed unrealistic from a condensed-matter perspective, during recent years, the Haldane (and related) models were realized experimentally using photon-dressed graphene [4], arrays of coupled waveguides [6], and periodically modulated optical lattices [35]. Again, it is of particular interest to understand the interaction effects in this model [36,37]. For the case of bosonic particles in the Haldane model, He et al. [38] have recently shown the emergence of a symmetry-protected bosonic integer quantum Hall phase by means of numerical simulations of large-scale cylinders. In Refs. [39] and [40], unconventional bosonic chiral superfluid phases have been found.

An important link between theory and the experimental realization of quantum-lattice gases with artificial gauge fields in the strongly correlated regime can be established by a reduction of the geometry from a two-dimensional model (which is typically theoretically challenging) to a two-or multi-leg ladder system. These quasi-one-dimensional models not only allow for an advanced theoretical treatment by means of powerful density-matrix renormalization group methods (DMRG) [41,42] or analytical bosonization techniques [43], but from the experimental perspective, they can be realized using various different implementations. Besides the superlattice method [13] and the use of digital mirror devices [44], various synthetic-lattice dimension approaches [14,15,45–48] have been employed. These use a coupling between internal states to realize some or even all lattice directions. While the theoretical interest in ladders with a flux dates back to
early studies of Josephson junction arrays [49–52], which was then extended to the strongly interacting regime in a seminal paper by Orignac and Giamarchi [53], the prospects of experimental realizations with ultracold quantum gases have led to tremendous theoretical activity. In particular, during the past years, the study of the low-dimensional relatives of, for example, the Hofstadter-Harper model on two- or three-leg ladder geometries has attracted a large deal of interest [53–81]. While fermionic systems are equally interesting [54,55], much work has focused on the ground-state phase diagram of bosonic systems, observing a multitude of phases resulting from the kinetic frustration due to the presence of a homogeneous flux per plaquette. These include Meissner phases characterized by a uniform edge current as well as commensurate and incommensurate vortex-fluid phases [53,56,67,82]. These phases can be characterized by the behavior of the chiral edge current and bulk currents or are distinct by the spontaneous breaking of a discrete symmetry (see Ref. [76] for an overview). We will refer to the ladders that result from the thin-cylinder limit of the Hofstadter-Harper model as flux ladders. Recent work addresses the possibility of stabilizing low-dimensional relatives of fractional quantum Hall states in ladder systems [63,64,78,79,83,84].

In this paper we study the ground-state physics of the bosonic Haldane model on a two-leg ladder geometry, which exhibits a rich physics. We will focus our analysis on the low filling regime of less than one particle per unit cell. We start our analysis with a description of the free-fermion version of the model (1), which allows us to understand some of the ground-state phases, and compare to the properties of hard-core bosons. We will refer to the ladders that result from the thin-cylinder limit of the Hofstadter-Harper model as flux ladders. Recent work addresses the possibility of stabilizing low-dimensional relatives of fractional quantum Hall states in ladder systems [63,64,78,79,83,84].

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which is one of the main results of this paper. For the case of finite interactions and in Sec. IV, we develop a weak-coupling picture of the emergence of the VL$_{1/2}$ phase, which we compare to numerical simulations. Finally, we discuss the ground-state phase diagram of hard-core bosons as a function of the phase $\phi_H$ for a fixed amplitude $J_H$ in Sec. V and conclude with a brief summary of our results presented in Sec. VI.

II. SINGLE-PARTICLE SPECTRUM AND FREE FERMIIONS

We start our analysis of the model from the free-fermion limit, which allows us to derive an initial picture of some of the liquid phases found also for bosons.

We express the Hamiltonian (1) in momentum space as

$$H_H = \sum_{k} \epsilon_k \tilde{c}_k \tilde{c}_k^\dagger.$$  

Here the momentum-space representations of the annihilation operators of the unit cell are grouped into a single vector \(\tilde{c}_k\) with \(\tilde{c}_k = \sum_{\gamma} e^{i k \gamma} c_{\gamma,k}\) and

\[
\hat{H}(k) = \begin{pmatrix} 2J_H \cos(2k + \phi_H) & J & (1 + e^{2ik})J \\ J & 2J_H \cos(2k - \phi_H) & 2J_H e^{-ik} \cos(k + \phi_H) \\ (1 + e^{2ik})J & 2J_H e^{ik} \cos(k + \phi_H) & 0 \end{pmatrix}.
\]

This can readily be diagonalized, leading to $H_H = \sum_{k=x,y,z=0}^{1,2} \epsilon_k \tilde{c}_k \tilde{c}_k^\dagger$ with new operators $\tilde{e}_k$, $\tilde{a}_{\gamma,k}$ with swapped positions of the $\tilde{a}_{\gamma,k}$ living on four generally separated energy bands $\epsilon_\gamma(k)$ with band index $\nu = 1,2,3,4$.

For $\phi_H \approx \pi$, in particular, we find a rich band structure $\epsilon_\gamma(k)$. For concreteness and unless stated otherwise, we fix the value of the phase to $\phi_H = 0.95\pi$. Since this is slightly detuned from $\phi_H = \pi$, there are finite chiral currents. We then vary the ratio $J_H/J$ as a free parameter.

Figure 3 shows four examples of the single-particle spectrum for various values of $J_H/J$ and $\phi_H$ for which different kinds of lowest band minima are realized: a single minimum at $k = 0$ [Figs. 3(a) and 3(b)], a single minimum at $k = \pm\pi/2$ [Fig. 3(c)], or two degenerate minima at $k = \pm Q$ [Fig. 3(d)]. We may associate these situations to four different low-density ground-state phases--three Meissner-like phases $M_0, M_{\theta}, M_{\pi/2}$ and an incommensurate vortex-fluid phase (V), which we will discuss in the following.

We may best characterize the different phases by calculating their local current and density configurations. Due to the explicitly broken time-reversal symmetry of the Hamiltonian (1), quantities of interest are the typically nonvanishing local and average particle currents on nearest- and next-nearest-neighbor bonds. A local current $\mathcal{J}(\ell \rightarrow \ell')$ from site $\ell$ to site $\ell'$ can be derived from the continuity equation $\frac{1}{T} \frac{\partial n_\ell}{\partial t} = -\sum_{\ell' \in \gamma} \mathcal{J}(\ell \rightarrow \ell') - \sum_{\ell' \in \gamma} \mathcal{J}(\ell' \rightarrow \ell)$.

Examples of the different local current structures in the three Meissner-like states are shown in Figs. 2(a), 2(b) and 2(c). Although the data shown are computed for hard-core bosons, the corresponding low-filling free-fermion version of these states looks similar. Similar to the nearest-neighbor currents $\mathcal{J}(A_{\gamma,\ell} \rightarrow B_{\gamma,\ell})$ and $\mathcal{J}(B_{\gamma,\ell} \rightarrow A_{\gamma,\ell+1})$ in the same direction along the legs, we dub the three phases Meissner phase (M). The inner currents on the rungs $\mathcal{J}(A_{\gamma,\ell} \rightarrow A_{2,\ell})$ are strongly suppressed.

In order to make the “Meissner character” of the phases more evident, in Fig. 4 we display the same current configurations for the three different Meissner phases of Fig. 2 with swapped positions of the A sites, i.e., relabeling $A_{1,\ell} \rightarrow A_{2,\ell}$. In this notation the strongest current runs through the outer boundary of the ladder system, which is a characteristic signature of a Meissner phase [67].

We may define an average chiral current on the nearest-neighbor bonds as

\[
\bar{j}_c = -\frac{1}{L} \sum_{\ell} \left[ \mathcal{J}(A_{\gamma,\ell} \rightarrow B_{\gamma,\ell}) + \mathcal{J}(B_{\gamma,\ell} \rightarrow A_{\gamma,\ell+1}) \right].
\]
In order to take into account the inner currents running on the next-nearest-neighbor bonds, we also introduce the average current $j_A$ ($j_B$) that runs through an $A_1$ ($B_1$) site:

$$j_A = -\frac{1}{L} \sum_\ell [\mathcal{J}(A_{1,\ell} \rightarrow B_{1,\ell}) + \mathcal{J}(A_{1,\ell} \rightarrow A_{1,\ell+1})$$

$$+ \mathcal{J}(A_{1,\ell} \rightarrow B_{2,\ell}) + \mathcal{J}(A_{1,\ell} \rightarrow A_{2,\ell})]$$

and

$$j_B = -\frac{1}{L} \sum_\ell [\mathcal{J}(B_{1,\ell} \rightarrow A_{1,\ell+1}) + \mathcal{J}(B_{1,\ell} \rightarrow A_{2,\ell+1})].$$

We can understand $j_A$ as an observable that quantifies the average current that runs from one hexagon to the neighboring one, while $j_B$ quantifies the current circulating inside the hexagon.

In the four-site unit cell, the density difference between $A$ and $B$ sites,

$$\Delta n = \frac{1}{L} \sum_{\ell, \gamma=1,2} \{n_{A_{\gamma,\ell}} - n_{B_{\gamma,\ell}}\},$$

typically is nonzero. We will refer to $\Delta n$ as the density imbalance.

In Fig. 5, we show the ground-state phase diagram of free fermions as a function of density $\rho = N/L$ (up to one particle per unit cell) and the nearest-neighbor tunneling amplitude $J_H/J$ with extended regions of the $M_0$, $M'_0$, and $M_{\pi/2}$ phases. In Fig. 6, the current and the density imbalance for a cut through the phase diagram at low filling are depicted. In the $M_0$ phase, the local currents on the next-nearest-neighbor bonds all circulate in the clockwise direction, opposite to the (small) currents on the nearest-neighbor links. Due to this almost closed ring-current within the hexagon, $j_A$ approximately vanishes in this phase. In the $M'_0$ phase, the sign of the diagonal next-nearest-neighbor currents is flipped compared to the $M_0$ phase. Hence, also the sign of $j_B$ is inverted compared to the $M_0$ phase and we observe a finite interhexagon current $j_A < 0$. While in both the $M_0$ and $M'_0$ phases the chiral current on the outer nearest-neighbor bonds $j_c$ is strongly suppressed, the $M_{\pi/2}$ phase is characterized by a larger $j_c$. Furthermore, we find $j_B > 0$ and $j_A > 0$ opposite to the $M'_0$ phase.

The expectation value of $\Delta n$ may be used to further distinguish the $M_0$ and $M'_0$ phases from each other (as can also be inferred from the color code of the dispersion relations in Fig. 3). The $M_0$ and $M_{\pi/2}$ phases have $\Delta n > 0$, while $\Delta n < 0$ for the $M'_0$ phase.

For higher fillings (or for special parameters also in the dilute limit) one encounters the situation that more than one Fermi sea forms, either by occupying modes of an overlapping higher band or because a second local minimum of the same band gets occupied. The doubling of the number of Fermi points is reflected by a change of the central charge parameter.
from \( c = 1 \) to \( c = 2 \). Due to the correspondence with the flux-ladder case [53,67], we generally refer to these phases as vortex-fluid phases (V), since the local current structure for a system with open boundary conditions exhibits a strong oscillatory but incommensurate pattern. (See Sec. V for a discussion of the analogous vortex-fluid phase for the case of hard-core bosons.) Interestingly, for the parameters of Fig. 5 and at the crossing from the \( M_0 \) to the \( M_0' \) phase, a tiny region with a doubly degenerate lowest band minimum emerges (see the inset in Fig. 5).

For special parameters, Dirac-like points exist in which two bands touch with a linear dispersion relation (see the red dot in Fig. 5). While for interacting fermions, nontrivial effects might be expected, for the case of (interacting) bosons this feature plays no role since finite filling properties do not carry over from fermions to bosons. For the filling of one particle per unit cell, a trivial band-insulating state is realized for larger values of \( J_H / J \).

### III. Ground-State Phase Diagram for Hard-Core Bosons

In the following we move on to the case of an interacting, single-component gas of bosons on the Haldane ladder. We start with the case of hard-core bosons (\( U / J \to \infty \)), which is the simplest case from a numerical perspective due to its restricted local Hilbert space and provides a good starting point to investigate the effect of interactions.

#### A. Diagnostic tools

Since this model is no longer exactly solvable, we perform density-matrix renormalization group (DMRG) simulations [41,42,85] with open boundary conditions to study the ground-state physics of this model, keeping up to \( \chi = 1000 \) DMRG states. We consider various system sizes of odd numbers of rungs \( L \), such that we simulate systems with \( (L - 1) / 2 \) hexagons.

Apart from extracting various order parameters, the density imbalance, and local currents, our DMRG calculations allow us to study further interesting quantum information measures. For example, the block entanglement entropy \( S_{cN} = -\text{Tr}[\rho_l \ln \rho_l] \) for the reduced density matrix \( \rho_l \) of a subsystem of length \( L \) may be employed to extract the central charge from the so-called Calabrese-Cardy formula [86–89]:

\[
S_{cN} = \frac{c}{3} \ln \left( \frac{L}{\pi} \sin \frac{\pi l}{2} \right) + \cdots.
\]  

Phase transitions may also be detected in the finite-size scaling of the fidelity susceptibility [90]

\[
\chi_{FS}(J_H) = \lim_{\delta J_H \to 0} \frac{-2 \ln |\langle \Psi_0(J_H) | \Psi_0(J_H + \delta J_H) \rangle|}{(\delta J_H)^2},
\]  

with \( |\Psi_0\rangle \) being the ground-state wave function.

#### B. Phase diagram

In Fig. 7, the ground-state phase diagram of hard-core bosons is shown for the parameters of Fig. 5. While in the limit of a dilute lattice gas, the same sequence of ground-state phases as for the case of free fermions is observed; for larger fillings, the differences become more drastic since the incommensurate vortex-fluid phases are suppressed while a vortex-lattice (VL1/2) phase gets stabilized, which we will describe in the following Sec. III.C in more detail. The current and density structure of this unconventional VL1/2 phase is shown in Fig. 2(d).

In Fig. 8, we show several observables and chiral currents for a cut through the phase diagram at a fixed density. As already anticipated in the previous section, the three different Meissner-like phases \( M_0 \), \( M_0' \), and \( M_{\pi/2} \) show a behavior similar to the free-fermion case discussed earlier (see Fig. 6): \( M_0 \) and \( M_0' \) phases can be discriminated by the sign change of the free-fermion case discussed earlier (see Fig. 6): \( M_0 \) and \( M_0' \) phases can be discriminated by the sign change of the \( \Delta n \) and \( j_B \) observables. While the \( M_0 \) phase is characterized by \( j_A \approx 0 \) and \( j_B < 0 \), we observe \( j_A > 0 \) and \( j_B > 0 \) in the \( M_0' \) phase and opposite signs, \( j_A > 0 \) and \( j_B < 0 \), in the \( M_{\pi/2} \) phase. By means of our numerical simulations we cannot resolve any intermediate phase between the \( M_0 \) and \( M_0' \) phases at finite densities.

For certain commensurate fillings, namely, at \( \rho = 1 / 4 \) and for the various Meissner phases but also at \( \rho = 1 / 8 \) for the VL1/2 phase, a charge gap opens (horizontal thick line in

![FIG. 7. Phase diagram for hard-core bosons in the Haldane ladder for \( \phi_H = 0.95\pi \) as a function of density \( \rho \) and the next-nearest-neighbor hopping parameter \( J_H / J \). There are three Meissner-like phases \( M_0, M_0', M_{\pi/2} \) as well as a vortex-lattice phase VL1/2. At density \( \rho = 1 / 8 \), a Mott-insulating state exists in the range \( 0.75 < J_H / J < 1 \), while at \( \rho = 1 / 4 \), the system is in a Mott-insulating state for any \( J_H / J \).](image)

![FIG. 8. Cut through the phase diagram of hard-core bosons Fig. 7 at density \( \rho = 0.19 \), showing the currents \( j_A, j_A / 2, j_B \), and the density imbalance \( 4\Delta n \).](image)
TABLE I. Quantum phases of strongly interacting bosons on the two-leg Haldane ladder studied in this work. The three different Meissner phases, M0, M0′, and Mπ/2, the vortex-liquid (V), and the vortex lattice VL1/2 phases exist either atop superfluid (SF) or Mott-insulating (MI) states. For simplicity here we list the properties of only the SF phases. We also list characteristic properties (see the text for details) such as the central charge c, counting the number of gapless modes, the size q of the effective unit cell in the ground state (i.e., the number of hexagons), the average local rung current \( j^r_{\text{avg}} \) [Eq. (10)], and the charge-density order \( O_{\text{DW}} \) [Eq. (11)] in the thermodynamic limit. The statements corresponding to the average currents \( j_A \) and \( j_B \), and the average density difference between A and B sites \( \Delta n \) [Eq. (7)] for the three different Meissner phases should be understood as a heuristically observed tendency. \( k_{\text{max}} \) denotes the position of the largest maximum of the momentum distribution function \( n(k) \) [Eq. (12)]. For the vortex phase, this typically corresponds to some incommensurate value \( 0 < Q < \pi/2 \).

<table>
<thead>
<tr>
<th>( j_A )</th>
<th>( j_B )</th>
<th>( \Delta n )</th>
<th>( c )</th>
<th>( q )</th>
<th>( j^r_{\text{avg}} )</th>
<th>( O_{\text{DW}} )</th>
<th>( k_{\text{max}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>M0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>M0′</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Mπ/2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>V</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>( \pm Q )</td>
<td>0</td>
</tr>
<tr>
<td>VL1/2</td>
<td>1</td>
<td>2</td>
<td>&gt;0</td>
<td>&gt;0</td>
<td>0/\pi</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Fig. 7. With this, the sequence of phase transitions becomes very rich, since the M0, M0′, and Mπ/2 phases may be observed on both a superfluid (SF) and a gapped Mott-insulator (MI) background (see Table I). Contrary to the free-fermion case, we observe the opening of a charge gap at \( \rho = 1/4 \) for all values of \( J_H/J > 0 \). At filling \( \rho = 1/8 \) the MI phase is apparently confined to the region of the VL1/2 phase. From our calculations, we cannot exclude the possibility of a small surrounding region of M0-MI and Mπ/2-MI phases at filling \( \rho = 1/8 \). Further details of the gapped regions will be discussed below in Sec. V.

C. Vortex-lattice phase

Contrary to the vortex-fluid phases, the VL1/2-SF phase is a single-component phase with a central charge \( c = 1 \) and it exhibits a spontaneously broken translational and parity symmetry. Therefore, the effective unit cell is doubled, as can be seen in Fig. 2(d). An order parameter for the VL1/2 phase can be defined from its average local rung current:

\[
j^r_{\text{avg}} = \frac{1}{L} \sum_{\ell=0,...,L} |\mathcal{J}(A_{\ell,1} \rightarrow A_{\ell,2})|.
\]

As an example, we present the finite average rung current \( j^r_{\text{avg}} \) in this region in Fig. 9(a). As shown in the insets of Fig. 9(a), the scaling of \( j^r_{\text{avg}} \) close to the quantum critical points follows Ising scaling relations and the data points from several finite-system-size simulations can be collapsed onto one single curve. As can be seen from the local density pattern shown in Fig. 2(d), the VL1/2 phase exhibits a finite density oscillation between adjacent unit cells. Hence, we may define a charge-density-wave order parameter via

\[
O_{\text{DW}} = \frac{1}{L} \left| \sum_{\ell=0,...,L} (-1)^\ell [n_{A_{\ell,1}} + n_{B_{\ell,2}}] \right|.
\]

Our numerical calculations indicate that \( O_{\text{DW}} \) stays finite in the thermodynamic limit [see the data for \( O_{\text{DW}}(L) \) shown in Fig. 9(b)], and its L dependence indeed looks almost identical to the plot of \( j^r_{\text{avg}} \) (L) in Fig. 9(a).

A further indication of the Ising character of both the M0-to-VL1/2 as well as the VL1/2-to-Mπ/2 transition is the approximate linear divergence of the peak of the fidelity susceptibility \( \chi_{FS}/L \propto L \) with system size \( L \), as seen in Fig. 10 [91,92]. Contrary to that, the highly nonlinear scaling of the maximum of \( \chi_{FS}(J_H)/L \) close to the M0-to-M0′ transition with respect to system size \( L \) [see Fig. 9(a)] may indicate a first-order transition. The same appears to be the case for the M0′-to-Mπ/2 transition at low fillings. However, due to the finite resolution of our calculations, we cannot exclude the possibility of small intermediate phases.
We want to stress, however, some important differences: The flux-ladder vortex-diverges. In commensurate fillings the VL$_{1/2}$ phase does not exhibit a charge gap and the single-particle correlations decay algebraically (as can also be seen from the presence of sharp peaks in Fig. 11). We may hence understand this liquid VL$_{1/2}$ phase with charge-density ordering as another type of a lattice phase is characterized by sharp peaks in the quasimomentum. The VL$_{1/2}$ phase may be seen as an analog of the vortex-lattice phase VL$_{1/2}$/SF phase on flux ladders [53,76], or the so-called biased ladder phase (BLP) on two-leg flux-ladder systems [59,76], which is, however, again most stable for the case of small interactions $U \rightarrow 0$. We may shed light on the mechanism for the stabilization of the VL$_{1/2}$ phase by means of a simple Bogoliubov-like approximation. We start by projecting the interaction to the lowest band,

$$H_{\text{eff}} = \sum_k \epsilon_k(k)a_k^\dagger a_k + \frac{U}{2} \sum_{k,k',q} V_{k,k',q} a_k^\dagger a_{k+q} a_{k+q}^\dagger a_k,$$

(13)

with $V_{k,k',q,k_h,k_q} = \sum_{v=1,...,4} U_{k_h}^{\dagger} U_{k_q}^{\dagger} U_{k_h} U_{k_q}$, where $U_{k,v}$ is the $v$th eigenvector of the Hamiltonian matrix Eq. (3). As an approximation in the limit of $\rho U \rightarrow 0$, we assume a condensation of the bosons at $Q = 0$ for $J_H \lesssim J_H^* \approx 0.69 \ldots J$ or case of hard-core particles on two-leg flux ladders [67]. Here, however, we find the vortex-lattice phase even for $U/J \rightarrow \infty$. The zigzag ladder chiral phases, on the other hand, are best understood from the dilute limit $\rho \rightarrow 0$ [99], in which it can be connected to the presence of a twofold degenerate band minimum for an extended parameter range, where interactions may favor either a two-component phase or a single-component chiral phase with spontaneously broken symmetry between the two dispersion minima. A similar mechanism applies to the so-called biased ladder phase (BLP) on two-leg flux-ladder systems [59,76], which is, however, again most stable for the case of small interactions $U \rightarrow 0$. In the present case of the VL$_{1/2}$ phase on the Haldane ladder, the single-particle spectrum is degenerate only for a single point at $J_H = J_H^*$. As we discuss in the following section, one may understand this VL$_{1/2}$ phase, naively transferring from the free-fermion case, as a spontaneous breaking of an effective emergent degeneracy between $k = \pi/2$ and $k = 0$ modes due to the finite filling and interactions.

IV. FINITE INTERACTION STRENGTHS $U/J < \infty$

In the following we analyze the stability of the vortex-lattice phase VL$_{1/2}$ for the case of finite repulsive on-site interactions $U/J < \infty$. The main analytical and numerical results are summarized in the phase diagram of Fig. 12.

A. Limit of weak interactions

In the weak-interaction limit $\rho U \rightarrow 0$, we may shed light on the mechanism for the stabilization of the VL$_{1/2}$ phase by means of a simple Bogoliubov-like approximation. We start by projecting the interaction to the lowest band,

$$H_{\text{eff}} = \sum_k \epsilon_k(k)a_k^\dagger a_k + \frac{U}{2} \sum_{k,k',q} V_{k,k',q,k,k'} a_k^\dagger a_{k+q} a_{k+q}^\dagger a_k,$$

(13)

with $V_{k,k',q,k_h,k_q} = \sum_{v=1,...,4} U_{k_h}^{\dagger} U_{k_q}^{\dagger} U_{k_h} U_{k_q}$, where $U_{k,v}$ is the $v$th eigenvector of the Hamiltonian matrix Eq. (3). As an approximation in the limit of $\rho U \rightarrow 0$, we assume a condensation of the bosons at $Q = 0$ for $J_H \lesssim J_H^* \approx 0.69 \ldots J$ or
at $Q = \pi/2$ and with $\alpha_Q \approx \sqrt{N} + \tilde{a}_Q$. Using $a_Q^\dagger a_Q^\dagger a_Q a_Q \approx N^2 - 2N \sum_{k \neq Q} \tilde{a}_k^\dagger \tilde{a}_k$, we rewrite the Hamiltonian retaining only quadratic terms,

$$H_{\text{eff}} \approx \sum_k A(k)(\tilde{a}_k^\dagger \tilde{a}_{-k} + \tilde{a}_{-k}^\dagger \tilde{a}_k) + \sum_k \tilde{B}(k)(\tilde{a}_k^\dagger \tilde{a}_{-k}^\dagger \tilde{a}_k \tilde{a}_{-k}).$$

(14)

A standard Bogoliubov transformation $\beta_k = u_k \tilde{a}_k - v_k \tilde{a}_{-k}$ diagonalizes the effective model $H_{\text{eff}} = E_0 + \sum_k \omega(k) \beta_k^\dagger \beta_k$ with $\omega(k) = \sqrt{A(k)^2 - B(k)^2}$, where $E_0$ is the ground state energy. Examples of the Bogoliubov excitation spectra $\omega(k)$ for values of $J_H / J$ in the $M_0$ and the $M_{\pi/2}$ phases are shown in Fig. 13.

Starting at $U = 0$ from the $M_{\pi/2}$ phase, with increasing interaction $\rho U$, the second minimum of the dispersion relation decreases and at some critical value touches zero at $k = 0$, as shown in Fig. 13(b). At this point the solution becomes unstable and the approximation of a single condensate at $Q = \pi/2$ is no longer valid. A finite occupation of modes around $k \approx 0$ has to be taken into account. Hence, we may associate this point of instability with the formation of a phase with a strong interplay between 0 and $\pi/2$ modes, which for large values of $\rho U$ can be identified to be the VL$_{1/2}$ phase. Note that the VL$_{1/2}$ phase is characterized by three maxima in the quasimomentum distribution function at $k = 0, \pm \pi/2$. Interestingly, starting from the $M_0$ phase, the condensate at $Q = 0$ seems to be stabilized with increasing $U/\rho$ and the second local minimum at $k = 0$ vanishes upon increasing the interaction strength, as shown in Fig. 13(a).

**B. Comparison to DMRG results**

Although the Bogoliubov approach is a crude simplification, we nevertheless obtain a decent qualitative agreement for the phase boundaries of the VL$_{1/2}$ phase with our numerical DMRG results. We employ a cutoff for the occupation of bosons per site of typically $n_{\text{max}} = 4$ bosons for $U \gtrsim J$ and fillings $\rho < 1$. By comparison with larger and smaller cutoffs, we have ensured the independence of the numerical data on the cutoff for the quantities shown in this work.

The lines of instability of the weak-coupling Bogoliubov method (see Fig. 12) predict a linear opening of the VL$_{1/2}$ phase for $J > J_{\text{cH}}^\prime$ which is consistent with the numerical estimates obtained for a finite filling $\rho = 0.2$ and interaction strength $U \approx J$. Again, for the $M_0$-to-$M_{\pi/2}$ transition, we do not resolve any intermediate phase in our numerical simulations.

**V. PHASE DIAGRAM AS A FUNCTION OF $\phi_H$**

In the range of parameters of Fig. 7, we did not find two-component vortex-fluid (V) phases for interacting bosons. However, for different parameters, where a lowest band minimum exhibits a degeneracy [see Fig. 3(d)], we observe a bosonic vortex-fluid phase. In Fig. 14, we show the ground-state phase diagram for hard-core bosons as a function of the phase $\phi_H$ and the filling $\rho$ for $J_H = J$. At $\rho = 1/4$, there are two regions where the system is in a Mott-insulating state, as indicated by the thick horizontal lines.

We show in Fig. 14 the phase diagram for hard-core bosons in the Haldane ladder as a function of the phase $\phi_H$ and the filling $\rho$ for $J_H = J$. At $\rho = 1/4$, there are two regions where the system is in a Mott-insulating state, as indicated by the thick horizontal lines.

**FIG. 13.** Bogoliubov excitation spectrum $\omega(k)$ for several interaction strengths $\rho U$ for $\Phi_H = 0.95\pi$ and (a) $J_H = 0.6J$ and (b) $J_H = J$.

**FIG. 14.** Phase diagram for hard-core bosons in the Haldane ladder as a function of the phase $\phi_H$ and the filling $\rho$ for $J_H = J$. At $\rho = 1/4$, there are two regions where the system is in a Mott-insulating state, as indicated by the thick horizontal lines.
Ref. [76], both \( F(j_R) \) and the momentum distribution show a similar behavior.

The peak position of \( F(j_R) \) in Fig. 17 exhibits a sharp jump for \( \phi_H \approx 0.86\pi \), which we here identify as the V-to-M’_0 transition point. Close to this V-to-M’_0 boundary, the quasimomentum distribution of Fig. 16 becomes blurred. Interestingly, in this part of the M’_0 region, \( F(j_R) \) also exhibits a distinct peak at \( k > 0 \), i.e., finite (boundary-driven) oscillations of the rung currents can still be found. Similar incommensurate Meissner-like phases have been discussed in Ref. [76] and have been connected to a certain class of Laughlin precursor states [78,79] for the case of two-leg flux ladders. Indeed, the presence of such additional intermediate phases close to the V-to-M’_0 boundary in this model should be examined in future studies more in detail.

For commensurate fillings \( \rho = 1/8 \) and \( \rho = 1/4 \), we observe the opening of a charge gap for certain values of \( \phi_H \). This can be the best seen in the \( \rho(\mu) \) curves displayed in Fig. 18 for different values of \( \phi_H \), where small horizontal plateaus at fillings \( \rho = 1/4 \) and \( \rho = 1/8 \) indicate the MI regions. In Fig. 19(a), we show the extracted charge gap,

\[
\Delta_c = \frac{E_0(N - 1) - 2E_0(N) + E_0(N + 1)}{2},
\]

extrapolated to the thermodynamic limit \( L \to \infty \). Due to the effects of the open boundary conditions, the particle density corresponding to the MI plateau is slightly offset from commensurability, depending on the choice of parameters. We display the data for \( N = (L \pm 1)/2 \) particles, which corresponds to the largest finite-size value of \( \Delta_c \).

Therefore, again, we observe the M_0 and M’_0 states and here also the vortex-fluid phases in both the SF and the MI background. For the case of a V-MI phase, we expect the
presence of a gapless neutral excitation and, hence, a central charge $c = 1$. In Fig. 19(c), we show the extracted central charge from fits to the entanglement entropy. Examples for the entanglement entropy and its dependence on block size are shown in Fig. 20. The results are consistent with $c = 0$ in the $M_0$-MI and $M_0'$-MI phases, $c = 1$ in the V- and $M_0$-SF phases, and $c = 2$ in the V-SF phase.

The horizontal arrows in Figs. 16, 17, and 19 show the estimated extension of the MI phases. Due to the Berezinskii-Kosterlitz-Thouless nature of the Mott-insulator to superfluid phase transitions, we give only an approximate extension based on the extrapolation of the charge gap and the calculation of the central charge.

In Fig. 19(c), we plot the behavior of chiral currents and the density imbalance for a cut through the phase diagram Fig. 14 at the commensurate filling $\rho = 1/4$. Consistent with our previous observations, we also find the characteristic features of the Meissner phases: For the $M_0$ phase, we find $j_c$ and $j_A \approx 0$ and $j_B < 0$ as well as $\Delta n > 0$, while for the $M_0'$ phase we mainly observe opposite signs, $j_B > 0$ and $\Delta n < 0$.

VI. SUMMARY

In summary, we have systematically studied the ground-state phase diagram of interacting bosons (and free fermions) for the Haldane model on a minimal realization of a two-leg ladder. Our main result is the emergence of an exotic type of a vortex-lattice-like phase for interacting bosons even for hard-core interactions. The VL$_{1/2}$ phase exhibits a finite rung-current order parameter as well as a finite charge-density-wave ordering. Since it emerges both at commensurate fillings with a charge gap but also at a broad range of incommensurate fillings on a superfluid gapless background, in the latter case, the VL$_{1/2}$ phase can be understood as another example of a lattice supersolid, i.e., a liquid with charge-density ordering.

We conclude by pointing to future research directions related to this model. In particular, the presence of analogs of Laughlin precursor states discussed in Refs. [78] and [79] may be examined in the region between the vortex-fluid and Meissner phases in the future. Further possible extensions include the analysis of quantum phases in extended lattice geometries, such as three-leg ladders and simplified (i.e., no next-to-nearest-neighbor tunneling) brick-wall ladders with a
FIG. 20. Examples of the entanglement entropy $S_{\phi}$ for various values of $\phi_H$ and hard-core bosons ($J = J_H, \rho = 0.25$) corresponding to the $M_0$-MI ($\phi_H = 0.2\pi$), $M_0'$-MI ($\phi_H = 0.4\pi$), $M_0$-SF ($\phi_H = 0.6\pi$), and $V$-SF ($\phi_H = \pi$) phases. The black dash-dotted lines correspond to a fit to Eq. (8).

ACKNOWLEDGMENTS

We are grateful to L. Santos and T. Vekua for useful discussions and we are indebted to G. Roux for his helpful comments on a previous version of the manuscript. S.G. acknowledges support from the German Research Foundation DFG (Project No. SA 1031/10-1). F.H.-M. acknowledges support from the DFG (Research Unit FOR 2414) via Grant No. HE 5242/4-1. Simulations were carried out on the cluster system at the Leibniz University of Hanover, Germany. The work of F.H.-M. was performed in part at the Aspen Center for Physics, which is supported by the National Science Foundation through Grant No. PHYS-1607611. The hospitality of the Aspen Center for Physics is gratefully acknowledged.